Numerical Solution of Laplace's Eq.

$$\nabla^2 \phi = -\frac{\rho}{\epsilon} \qquad \text{(Poisson's Eq.)}$$

$$\nabla^2 \phi = 0 \qquad \text{(Laplace's Eq).}$$

notice that Laplace's equation is a homogeneous P.D.E. $\therefore$ $\phi = 0$ is automatically a solution. Where do the non-trivial solutions come from?

$\Big\{$ ANS: imposed boundary conditions determine the form of the solution.

example (1-D):

(O.D.E.) $\dfrac{d^2\phi}{dx^2} = 0 \qquad 0 \le x \le 1$

the <u>general solution</u> is

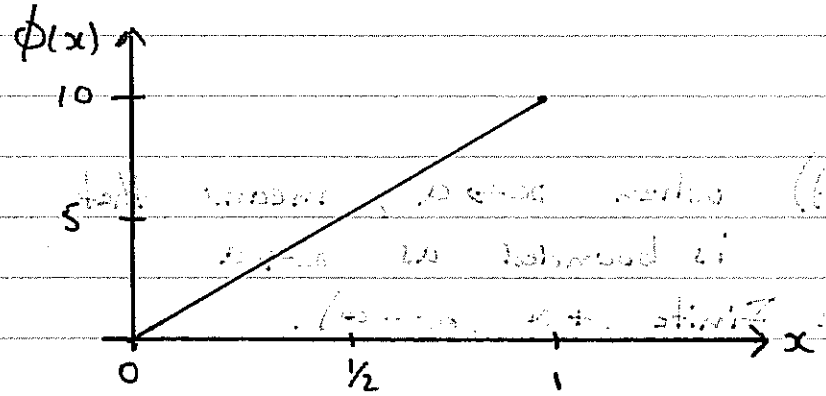
$$\phi(x) = k_1 x + k_2$$

k_1, k_2 - arbitrary constants.

(B.C.) $\phi(0) = 0$, $\phi(1) = 10$
(Dirichlet).

$\phi(0) = k_2 = 0 \qquad \phi(1) = 10 = k_1$

$$\therefore \boxed{\phi(x) = 10x}$$



satisfies both O.D.E + B.C.

How would we apply some numerical approximation to the boundary value problem (B.V.P)

$$\frac{d^2\phi}{dx^2} = 0 \qquad \phi(0) = 0 , \ \phi(1) = 10 \quad ?$$

Main idea in finite difference techniques is to approximate the fcn were looking for $\phi(x)$ by a grid fcn

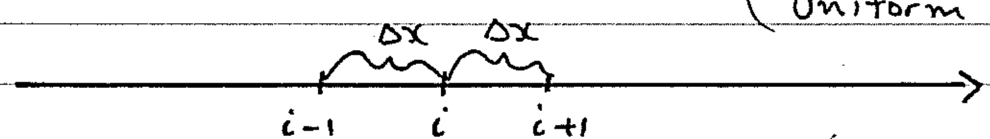
$$\phi_i \approx \phi(i\Delta x)$$

and then discretize the differential operators. (usually using Taylor's expansion).

Given the grid fcn $\quad f_i \approx f(i\Delta x)$

$(i \in \mathbb{Z})$

(uniform grid)



$(\Delta x > 0)$

Taylor's expansion

$$f_{i+1} = f_i + \Delta x f_i' + \frac{\Delta x^2 f_i''}{2} + \frac{\Delta x^3 f_i'''}{3!} + \dots \qquad (1)$$

solving for $f_i' \approx f'(i\Delta x)$

$$f_i' = \frac{f_{i+1} - f_i}{\Delta x} - f_i'' \frac{\Delta x}{2} - f_i''' \frac{\Delta x^2}{3!} - \dots \qquad (1')$$

Forward difference approximation: (f. d.)

$$\boxed{f_i' = \frac{f_{i+1} - f_i}{\Delta x} + O(\Delta x)}$$

$O(\Delta x)$ → "of order Δx"
→ truncation error due to replacing an continuous process by a discrete one.

$\boxed{\text{continuous derivative}} \rightleftharpoons \boxed{\text{Discrete finite difference.}}$

$(\Delta x > 0)$

$$f_{i-1} = f_i - \Delta x\, f_i' + \frac{\Delta x^2}{2} f_i'' - f_i''' \frac{\Delta x^3}{6} + \dots \qquad (2)$$

$$f_i' = \frac{f_i - f_{i-1}}{\Delta x} + \frac{\Delta x}{2} f_i'' - f_i''' \frac{\Delta x^2}{6} \qquad (2')$$

<u>backward difference</u> (b.d.)

$$\boxed{f_i' = \frac{f_i - f_{i-1}}{\Delta x} + O(\Delta x)}$$

now take eq. (1) - eq (2) :

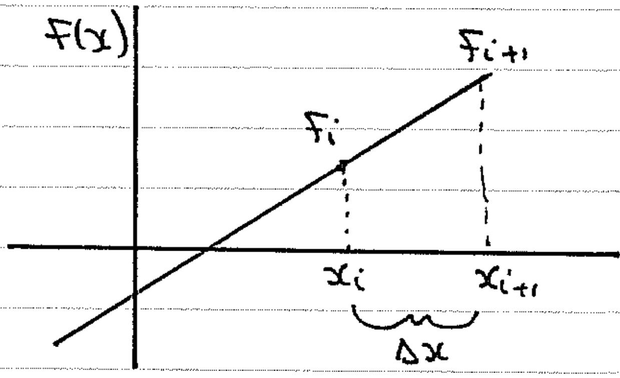
$$f_{i+1} - f_{i-1} = f_i'\, 2\Delta x + f_i''' \frac{\Delta x^3}{3} + \dots$$

$$f_i' = \frac{f_{i+1} - f_{i-1}}{2\Delta x} - f_i''' \frac{\Delta x^2}{6} + \dots$$

<u>central difference</u>

$$\boxed{f_i' = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + O(\Delta x^2)}$$

1) let us show that the f.d. and b.d. operators are exact for linear fcns.



$$f(x) = \left(\frac{f_{i+1} - f_i}{x_{i+1} - x_i}\right)(x - x_i) + f_i$$

$$\boxed{f'(x) = \frac{f_{i+1} - f_i}{\Delta x}}$$

(obvious - slope)

(For b.d. just relable $i+1 \rightarrow i$ $\quad i \rightarrow i-1$).

2) Show that the central difference operator is exact for quadratic fcns.

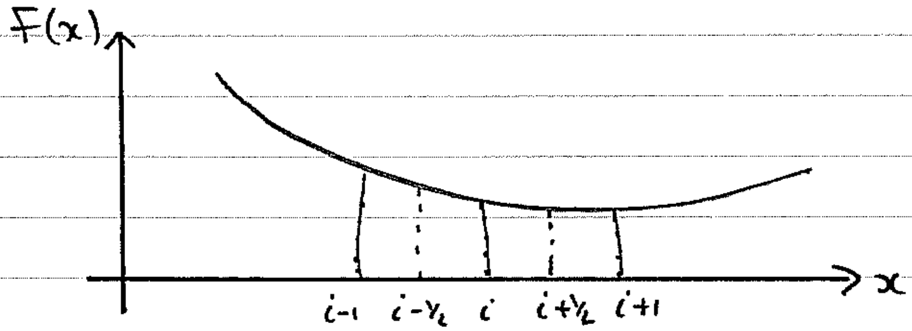
$$f_i' = \frac{f_{i+1} - f_{i-1}}{2\Delta x} - f_i''' \frac{\Delta x^2}{6} + \cdots$$

if $f(x) = a + bx + cx^2$ (arbitrary quadratic).

$$\Longrightarrow f'''(x) = 0$$

$\therefore$ c.d. exact.

approximations for higher order derivatives:



using central differences at $i-\frac{1}{2}$, $i+\frac{1}{2}$:

$$f'_{i+\frac{1}{2}} = \frac{f_{i+1} - f_i}{\Delta x} + O(\Delta x^2)$$

$$f'_{i-\frac{1}{2}} = \frac{f_i - f_{i-1}}{\Delta x} + O(\Delta x^2)$$

$$\therefore \quad f''_i = \frac{f'_{i+\frac{1}{2}} - f'_{i-\frac{1}{2}}}{\Delta x} + O(\Delta x^2)$$

$$= \frac{\frac{f_{i+1} - f_i}{\Delta x} - \frac{f_i - f_{i-1}}{\Delta x}}{\Delta x} + O(\Delta x^2)$$

$$\boxed{f''_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} + O(\Delta x^2)}$$

Note it turns out that the two $O(\Delta x^2)$ errors cancel to see this use

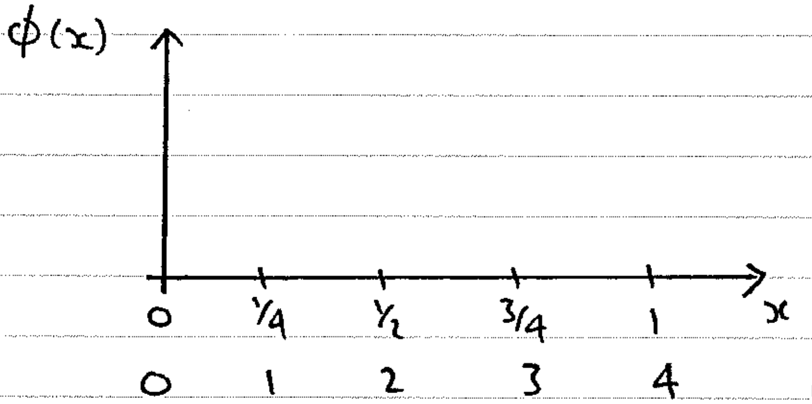
① − ② !
solve for f''

Now we can apply this to Laplace's eq in 1-D: $(h = \Delta x)$

$$\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} = 0$$

$$\phi_{i+1} - 2\phi_i + \phi_{i-1} = 0$$

let $h = 1/4$



B.C.'s.

$\phi_0 \approx \phi(0\Delta x) = 0$

$\phi_4 \approx \phi(4 \tfrac{1}{4}) = \phi(1) = 10$

$$\left.\begin{aligned} \phi_0 - 2\phi_1 + \phi_2 &= 0 \\ \phi_1 - 2\phi_2 + \phi_3 &= 0 \\ \phi_2 - 2\phi_3 + \phi_4 &= 0 \end{aligned}\right\}$$ three equations from applying c.d. at 3 internal points.

3 unknowns ϕ_1, ϕ_2, ϕ_3.

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$ ← From B.C.

by Gaussian ellimination:

$$\phi_1 = \frac{5}{2} \qquad \phi_2 = 5 \qquad \phi_3 = \frac{15}{2}$$

recall exact solution: $\phi(x) = 10x$

$$\begin{cases} \phi(\tfrac{1}{4}) = 10(\tfrac{1}{4}) = \dfrac{5}{2} = \phi_1 \\ \phi(\tfrac{1}{2}) = 10(\tfrac{1}{2}) = 5 = \phi_2 \\ \phi(\tfrac{3}{4}) = 10(\tfrac{3}{4}) = \dfrac{15}{2} = \phi_3 \end{cases}$$

$\therefore$ the second order approximation gives the exact solution on the grid points. (as it should since exact sol^n is linear).

Try for more grid points!

Now what have we solved for? "ϕ"

This is like a nodal voltage.

$$\bar{E} = -\frac{d\phi}{dx}\,\hat{a}_x = -10\,\hat{a}_x$$

if $\phi(0) = 10$ $\quad \phi(1) = 20 \Rightarrow \phi(x) = 10x + 10$

$$\bar{E} = -10\,\hat{a}_x \quad \text{still!}$$

$\bar{E}$ Field or voltage differences is what we can measure physically not ϕ!

Other types of Boundary Conditions

Say we were given E at one boundary.

$$-\frac{d\phi}{dx} = \bar{E} = 10 \quad @ \quad x = 1$$

$N-1 \quad N \quad N+1$

↑ boundary.

$$-\frac{d\phi}{dx} \approx \frac{\phi_{N-1} - \phi_{N+1}}{2\Delta x} = 10$$

$O(\Delta x^2)$

$\therefore$ at the point N:

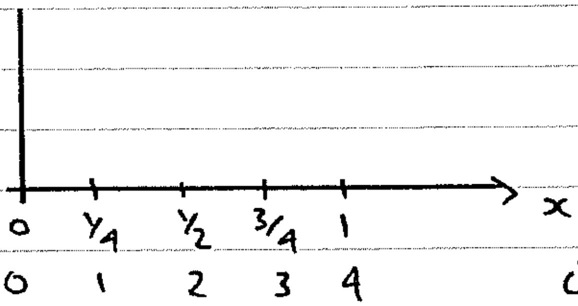
$$\frac{d^2\phi}{dx^2} \simeq \frac{\phi_{N-1} + \phi_{N+1} - 2\phi_N}{\Delta x^2} = 0$$

we don't wan't to introduce new node to solve for "ϕ_{N+1}"

$$\therefore \text{ use } \quad \phi_{N+1} = \phi_{N-1} - 20\Delta x$$

$$\therefore \quad \phi_{N-1} + \phi_{N-1} - 20\Delta x - 2\phi_N = 0$$

$$\boxed{2\phi_{N-1} - 20\Delta x - 2\phi_N = 0}$$



$\phi_0 = 0$

$\Delta x = \frac{1}{4}$

@ $i=1$ $\phi_0 - 2\phi_1 + \phi_2 = 0$

@ $i=2$ $\phi_1 - 2\phi_2 + \phi_3 = 0$

@ $i=3$ $\phi_2 - 2\phi_3 + \phi_4 = 0$

@ $i=4$ $2\phi_3 - 5 - 2\phi_4 = 0$

$$\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

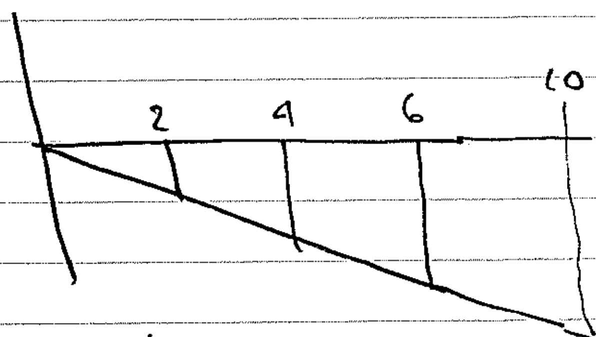
$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -3/2 & 1 & 0 & 0 \\ 0 & 0 & -5/3 & 1 & 0 \\ 0 & 0 & 2 & -2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & -3/2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 3/5 & 0 \\ 0 & 0 & 0 & -4/5 & 5 \end{bmatrix}$$

$\phi_4 = -10$

$\phi_3 = -6$

$\phi_2 = -4$

$\phi_1 = -2$



check that this is exact.

$\phi(x) = k_1(x) + k_2$ $\phi_0 = 0 = k_2$ $\left.\frac{d\phi}{dx}\right|_{x=1} = -10 = k_1$ $\boxed{\phi(x) = -10x}$

Application of Iterative Methods to Solve Matrix Equations

take same example as before:

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix}$$

solve for diagonal components:

$$\left.\begin{aligned} -2\phi_1 &= -\phi_2 &\Rightarrow\quad \phi_1^{(k)} &= \tfrac{1}{2}\phi_2^{(k-1)} \\ -2\phi_2 &= -\phi_1 - \phi_3 &\Rightarrow\quad \phi_2^{(k)} &= \tfrac{1}{2}\left(\phi_1^{(k-1)} + \phi_3^{(k-1)}\right) \\ -2\phi_3 &= -\phi_2 - 10 &\Rightarrow\quad \phi_3^{(k)} &= \tfrac{1}{2}\phi_2^{(k-1)} + 5 \end{aligned}\right\}$$

K	$\phi_1^{(k)}$	$\phi_2^{(k)}$	$\phi_3^{(k)}$	$\|\delta\| = \Sigma\|\delta_i\|$
0	1.00	1.00	1.00 guess	0
1	0.5	1.0	5.5	5
2	0.5	3.0	5.5	2
3	1.5	3.0	6.5	2
4	1.5	4.0	6.5	1
5	2.0	4.0	7.0	1
⋮	⋮	⋮	⋮	⋮
14	2.46875	4.96875	7.46875	0.03125
exact:	2.5	5.0	7.5	

A). This is called <u>Jacobi's method</u> is usually slow to converge.

in <u>matrix form</u> : $Ax = b$

then choose D to be the diagonal of A

$$\boxed{D x^{(K+1)} = (D-A)\, x^{(K)} + b}$$

in <u>algebraic form</u>: $[a_{ij}]_{n \times n} (x_i)_n = (b_i)_n$

$$\boxed{(x_i)^{(K+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{(K)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(K)}}$$

B) Gauss-Seidel Method (G-S)

<u>succinctly</u>: use the new $(K+1)$ values of x as you find them!

same example:

$$\left.\begin{aligned}
-2\phi_1 &= -\phi_2 &\Rightarrow\quad \phi_1^{(K+1)} &= \tfrac{1}{2}\phi_2^{(K)} \\
-2\phi_2 &= -\phi_1 - \phi_3 &\Rightarrow\quad \phi_2^{(K+1)} &= \tfrac{1}{2}\left(\phi_1^{(K+1)} + \phi_3^{(K)}\right) \\
-2\phi_3 &= -\phi_2 - 10 &\Rightarrow\quad \phi_3^{(K+1)} &= \tfrac{1}{2}\phi_2^{(K+1)} + 5
\end{aligned}\right\}$$

K	$\phi_1^{(K)}$	$\phi_2^{(K)}$	$\phi_3^{(K)}$	$\lVert\delta\rVert = \Sigma\lvert\delta_i\rvert$
0	1.00	1.00	1.00	0
1	0.5	0.75	5.375	5.125
2	0.375	2.875	6.4375	3.3125
⋮	⋮	⋮	⋮	⋮
10	2.4916993	4.9916992	7.4958496	~0
exact:	2.5	5.0	7.5	

in <u>matrix form</u> : $Ax = b$

choose E to be the <u>lower triangular</u> part of A (including diagonal)

$$\boxed{E\, x^{(k+1)} = (E-A)\, x^{(k)} + b}$$

in <u>algebraic form</u>: $[a_{ij}]_{n\times n} (x_i)_n = (b_i)_n$

$$\boxed{(x_i)^{(k+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{(k+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(k)}}$$

This method is still relatively slow to converge.

c) Successive Overrelaxation (SOR)

<u>succinctly</u>: take the difference between the G-S calculated point and the original value of the point and multiply by the <u>over-relaxation</u> factor $\boxed{1 < \omega < 2}$ then add to the original value.

same example:

$$-2\phi_1 = -\phi_2 \Rightarrow \quad \phi_1^{(k+1)} = \omega\left[\tfrac{1}{2}\phi_2^{(k)} - \phi_1^{(k)}\right] + \phi_1^{(k)}$$

$$-2\phi_2 = -\phi_1 - \phi_3 \Rightarrow \quad \phi_2^{(k+1)} = \omega\left[\tfrac{1}{2}\left(\phi_1^{(k+1)} + \phi_3^{(k)}\right) - \phi_2^{(k)}\right] + \phi_2^{(k)}$$

$$-2\phi_3 = -\phi_2 - 10 \Rightarrow \quad \phi_3^{(k+1)} = \omega\left[\tfrac{1}{2}\phi_2^{(k+1)} + 5 - \phi_3^{(k)}\right] + \phi_3^{(k)}$$

in *matrix form*: $Ax = b$

choose M to be the lower triangular part of A as in G-S but divide the diagonal entries by ω (overrelaxation factor)

$$\boxed{M x^{(k+1)} = (M-A)\, x^{(k)} + b}$$

in *algebraic form*: $[a_{ij}]_{n\times n} (x_i)_n = (b_i)_n$

$$\frac{a_{ii}}{\omega} (x_i)^{(k+1)} = b_i - \sum_{j=1}^{i-1} a_{ij} (x_j)^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} (x_j)^{(k)} + \left(\frac{1}{\omega} - 1\right) a_{ii} (x_i)^{(k)}$$

$$(x_i)^{(k+1)} = \omega \overbrace{\left[\frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} (x_j)^{(k+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} (x_j)^{(k)}\right]}^{x_{i\,G-S}^{(k+1)}} + (1-\omega)(x_i)^{(k)}$$

$$\boxed{(x_i)^{(k+1)} = \omega\left[(x_i)^{(k+1)}_{G-S} - (x_i)^{(k)}\right] + (x_i)^{(k)}}$$

For optimum "ω" this method is usually much faster to converge.

we find this optimum "ω" experimentally or check theory for specific P.D.E's.

Stability of Iterative Methods

consider the linear equation

$$A x = b \qquad (1)$$

and an arbitrary simpler matrix M which is inserted into (1) as:

$$M x + A x = M x + b$$
$$M x = (M - A) x + b \qquad (2)$$

now use (2) to develop an iterative scheme starting from an initial guess x^0 :

$$M x^{k+1} = (M - A) x^k + b \qquad (3)$$

now if the series $x^0, x^1, x^2, \ldots$ converges to say x^∞ then

$$M x^\infty = (M - A) x^\infty + b$$

$$\Rightarrow \quad A x^\infty = b$$

$\therefore$ $\boxed{x = x^\infty}$ is the solution of (1) the convergence of the series to the "fixed point" x^∞ depends on the properties of A and the chosen M. (note: M has been arbitrary until now).

if we subtract ③ from ② we can define the error at step K as

$$Mx - Mx^{K+1} = (M-A)x - (M-A)x^K + b - b$$
$$M(x - x^{K+1}) = (M-A)(x - x^K)$$

$$e^K = x - x^K \qquad \text{error at } K^{th} \text{ step}$$

$\therefore$ error equation: $\quad M\, e^{K+1} = (M-A)\, e^K \qquad ④$

this is a one-step difference equation and the R.H.S of the original system (b) is not involved,

$$e^{K+1} = M^{-1}(M-A)\, e^K = B\, e^K \qquad ⑤$$

$\therefore$ at each new iteration the current error vector is multiplied by the matrix B

If e^0 is the error in the initial guess then we can write:

$$e^K = B^K e_0 \qquad ⑥$$

thus we see from ⑥ that

if $B^k \to 0$ as $k \to \infty$ then
$e^k \to 0$ as $k \to \infty$
and $\therefore$ $x^k \to x$

$\therefore$ To test that our iterative scheme will converge we must test that

$B^k \to 0 \implies$ that our difference equation is <u>stable.</u>

A useful theorem for our purposes is the following:

<u>def^n</u>: spectral radius, $\rho(A)$, of a matrix A is its largest eigenvalue, in magnitude:

$$\rho(A) = \max_{\lambda \in \Lambda(A)} |\lambda|$$

wher $\Lambda(A)$ is the set of eigenvalues of the matrix A.

<u>Theorem</u>:
The powers B^k approach zero iff

$$\rho(B) < 1$$

The rate of convergence is governed by the size of $\rho(B)$

$|\lambda_i| < 1 \Rightarrow$ that the eigenvalues must lie inside the unit circle in the complex plane.

<u>Examples</u>: check convergence of the following:

a) $B = \begin{bmatrix} 1/2 & 10 \\ 0 & 1/2 \end{bmatrix}$ $\quad (\lambda - 1/2)(\lambda - 1/2) = 0$

$\lambda_1 = \frac{1}{2} \quad \lambda_2 = 1/2 \Rightarrow$ convergence

b) $B = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ $\quad (\lambda - 1/2)(\lambda - 1/2) - 1/4 = 0$

$\lambda^2 - \lambda + 1/4 - 1/4 = 0$

$\lambda(\lambda - 1) = 0$

$\lambda = 0, \ \lambda = 1$

$\Rightarrow$ divergent

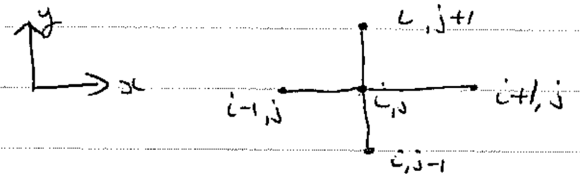
in fact

$$B^k = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

2-D Laplace Equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi^2}{\partial y^2} = 0$$

computational molecule:

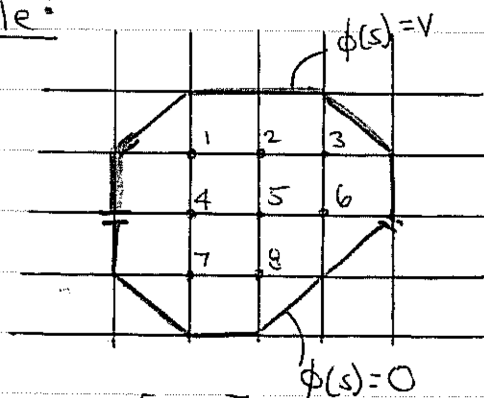


$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \approx \frac{\phi_{i+1,j} - 2\phi_{ij} + \phi_{i-1,j}}{(\Delta x)^2} + \frac{\phi_{i,j+1} - 2\phi_{ij} + \phi_{ij-1}}{(\Delta y)^2}$$

if $\Delta x = \Delta y = h$ (i.e square grid)

$$\bar{\nabla}^2\phi \approx \frac{\phi_{i+1j} + \phi_{i-1,j} + \phi_{ij+1} + \phi_{ij-1} - 4\phi_{ij}}{h^2} = 0$$

example:



$$A = \begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix}$$

$$\underline{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \\ \phi_8 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} -2V \\ -V \\ -2V \\ -V \\ 0 \\ -V \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{A\underline{\phi} = \underline{b}}$$

"A" will be a sparse matrix and so iteration techniques are appropriate. Notice that A will be diagonally dominant which is good for convergence.

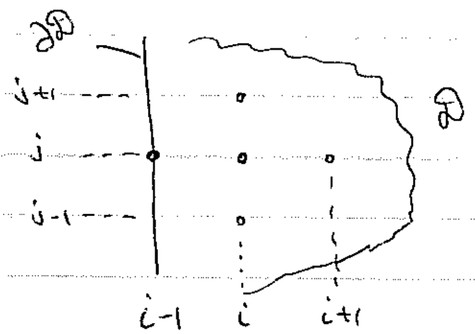
In two space dimensions it is more convenient to lable the nodes with their appropriate double indicies ϕ_{ij}

and as long as all boundary points lie on the "integer" locations then we can use

$$\phi_{i+1j} + \phi_{i-1j} + \phi_{ij+1} + \phi_{ij-1} - 4\phi_{ij} = 0$$

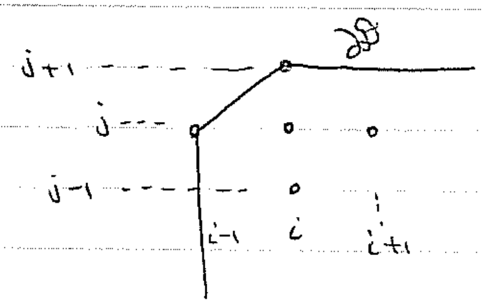
at all the interior points of the domain as well as the ones which have neighbouring boundaries in which the value of ϕ is specified

Dirichlet boundary conditions.



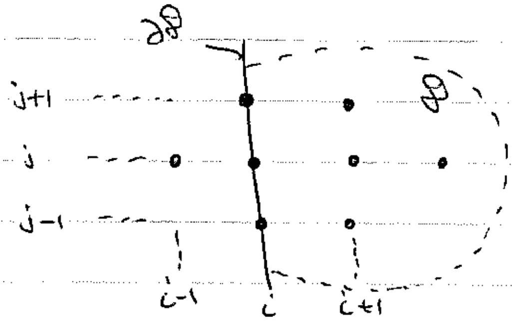
if on $\partial\mathcal{D}$ $\phi = V$, say, then

$$\phi_{i+1j} + V + \phi_{ij+1} + \phi_{ij-1} - 4\phi_{ij} = 0$$



$$\phi_{i+1j} + V + V + \phi_{ij-1} - 4\phi_{ij} = 0$$

now if the normal derivative is specified at a boundary point:



if on $\partial\mathcal{D}$ $\frac{\partial \phi}{\partial n} = 0$ $\left(\frac{\partial \phi}{\partial x} = 0\right)$

(Neumann boundary condition)

then we can consider a fictitious grid point to the left of the boundary which does not belong in $\mathcal{D}$.

Laplace's equation at $(i\,j)$ gives.

$$\phi_{i+1\,j} + \phi_{i-1\,j} + \phi_{i\,j+1} + \phi_{i\,j-1} - 4\phi_{i\,j} = 0 \qquad (1)$$

Neumann's B.C. at $(i\,j)$ expressed as a central difference approximation gives:

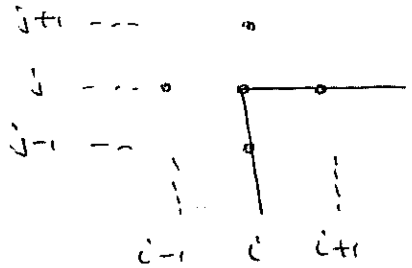
$$\frac{\phi_{i+1\,j} - \phi_{i-1\,j}}{2h} = 0 \qquad O(h^2)$$

$$\text{or} \quad \phi_{i+1\,j} = \phi_{i-1\,j} \qquad (2)$$

using (2) in (1) gives:

$$\boxed{2\phi_{i+1\,j} + \phi_{i\,j+1} + \phi_{i\,j-1} - 4\phi_{i\,j} = 0} \qquad (3)$$

at a corner with two Neumann B.C's we get:



$$\phi_{i,j+1} = \phi_{i,j-1} \quad , \quad \phi_{i+1,j} = \phi_{i-1,j}$$

$$\boxed{2\phi_{i+1,j} + 2\phi_{i,j-1} - 4\phi_{ij} = 0} \quad (4)$$

Now that we have discussed the special cases of the boundary conditions, we can formulate the SOR scheme:

at a regular point:

$$\phi_{ij} = \frac{\omega}{4}\left[\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1}\right] + (1-\omega)\phi_{ij}$$

at an interior point bounded by a Dirichlet point:

$$\phi_{ij} = \frac{\omega}{4}\left[\phi_{i+1,j} + V + \phi_{i,j+1} + \phi_{i,j-1}\right] + (1-\omega)\phi_{ij}$$

etc.

<u>Notice</u> we are using the most upto date ϕ_{ij}'s and over-writting ϕ_{ij}.

$\therefore$ in implementation only one array $\phi(n,n)$ is required

* odd-even updating.

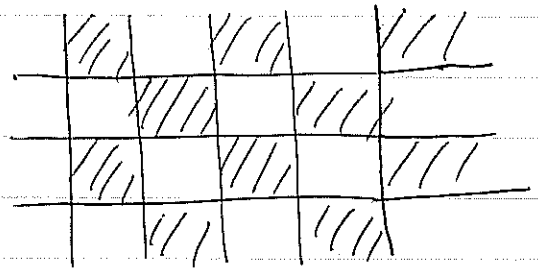
For the Laplace equation notice that the updating of a point (i,j) involves only the points $(i+1,j)$ $(i-1,j)$ $(j,i+1)$ $(j,i-1)$

Thus we can update all points with

$$\boxed{(i+j) \bmod 2 = 0}$$

First and then scan through the points with $\boxed{(i+j) \bmod 2 \neq 0}$

(checker board effect)



Stopping condition:

we define a <u>displacement norm</u> as.

$$\delta = \sum_{i=1}^{N} \left| \phi_i^{(m+1)} - \phi_i^{(m)} \right| = \|\Delta\phi\|$$

and a relative displacement norm as

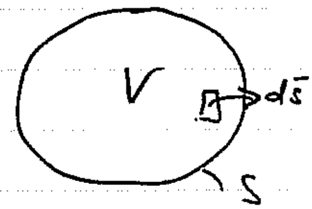
$$\epsilon = \frac{\delta}{\|\underline{\phi}\|} = \frac{\delta}{\sum_{i=1}^{n} |\phi_i^{(m+1)}|}$$

we stop when the calculated error is below a specified ϵ (say 10^{-8})

Integration of Electric Flux Density

From Gauss' Law we know that

$$\Phi_D = \oiint_S \bar{D} \cdot d\bar{s} = \iiint_V q \, dv = Q_T$$

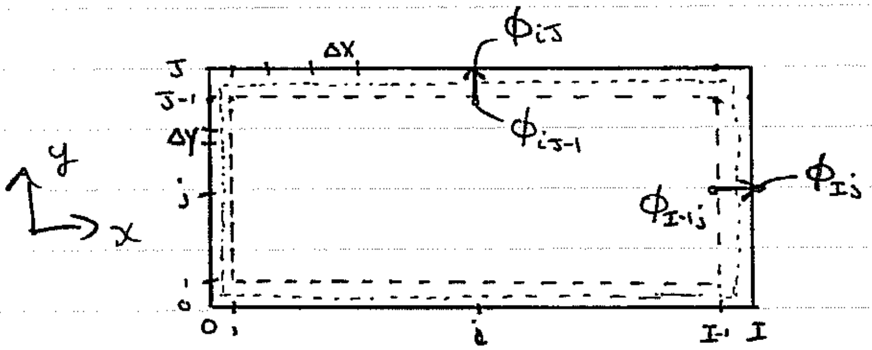


For the assigned problem: $Q_T = 0$

$$\therefore \oiint_S \bar{D} \cdot d\bar{s} = 0$$

and we have $\epsilon = \epsilon_0$ inside the volume

$$\therefore \oiint_S \bar{E} \cdot d\bar{s} = 0$$

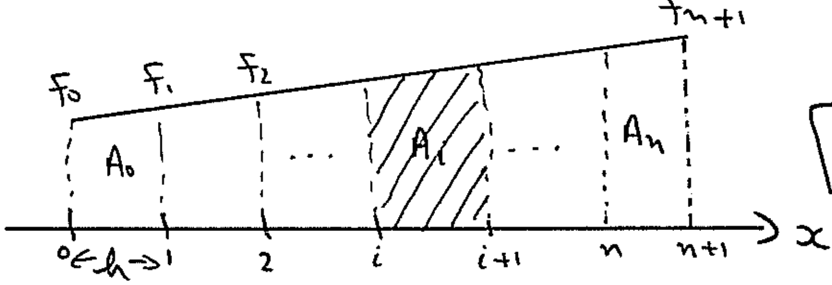


$$E_{y_{iJ-\frac{1}{2}}} \approx \frac{\phi_{iJ} - \phi_{iJ-1}}{\Delta y} + O(\Delta y^2$$

$$E_{x_{I-\frac{1}{2}j}} \approx \frac{\phi_{Ij} - \phi_{I-1j}}{\Delta x} + O(\Delta x^2$$

$$\oiint_S \bar{E} \cdot d\bar{s} = -\int_{j=0}^{j=J} E_{x_{0j}} dy + \int_{i=0}^{i=I} E_{y_{iJ}} dx + \int_{j=0}^{j=J} E_{x_{Ij}} dy - \int_{i=0}^{i=I} E_{y_{i0}} dx$$

Numerical integration:



Trapezoidal Rule

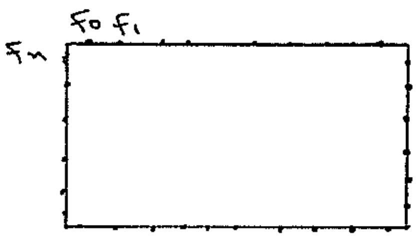
$$\boxed{A_i \approx \tfrac{1}{2}(f_i + f_{i+1})h}$$

$$I = \int f(x)\,dx = \sum_{i=0}^{n} \tfrac{1}{2}(f_i + f_{i+1})\,h + O(h)$$

$$\boxed{I \approx h\left(\tfrac{1}{2}f_0 + f_1 + \cdots f_n + \tfrac{1}{2}f_{n+1}\right)}$$

Around a closed loop:

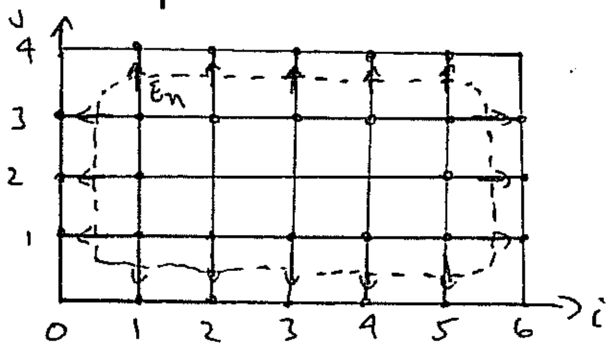
$f_0 = f_{n+1}$



$$\therefore\ I \approx (f_0 + f_1 + \cdots\ f_n)h$$

$$I \approx \left(\sum_{i=0}^{n} f_i\right) h$$

For our case the f are the normal component of electric field. $(h = \Delta x = \Delta y)$



$$\Phi_D \approx \sum_{i=1}^{5}\overbrace{\left(\frac{\phi_{i4} - \phi_{i3}}{h}\right)}^{E_n} h + \sum_{i=1}^{5}\left(\frac{\phi_{i0} - \phi_{i1}}{h}\right) h + \sum_{j=1}^{3}\left(\frac{\phi_{0j} - \phi_{1j}}{h}\right) h + \sum_{j=1}^{3}\left(\frac{\phi_{6j} - \phi_{5j}}{h}\right) h$$

$\Phi_D \approx$ (Sum outside potentials)
− (Sum inside potentials + inside corners)

$\approx$ $\underline{\underline{\text{zero}}}$

This should be checked in the program.

say $i = 0(1)6$ $j = 0(1)6$ i.e. square.

```
flux = 0
For j = 1(1)5
    (outer boundary)
    flux = flux + φ(0,j)          Left
    flux = flux + φ(j,0)          Bottom
    flux = flux + φ(6,j)          Right
    flux = flux + φ(j,6)          Top

    (inner boundary)
    flux = flux − φ(1,j)          Left
    flux = flux − φ(j,1)          Bottom
    flux = flux − φ(5,j)          Right
    flux = flux − φ(j,5)          Top.
end For
```